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COST AND PRODUCTION FUNCTIONS

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The characterization of external effects as "separable" has played an important role in the development of the theory of externalities. The separable case is particularly well behaved when procedures for achieving an optimum allocation of resources in the presence of externalities are examined. Davis and Whinston (1962) find that separability assures the existence of a certain kind of equilibrium in bargaining between firms which create externalities, and that equilibrium does not exist without separability. Kneese and Bower (1968) argue that with separability the computation of Pigovian taxes to remedy externalities is particularly simple. Marchand and Russell (1974) demonstrate that certain liability rules regarding external effects lead to Pareto optimal outcomes if and only if externalities are separable. In each of these cases the problem is posed in terms of two firms related by technological externalities, and separability is defined in terms of a cost function. In this paper we will characterize that class of production functions which give rise to separable cost functions, and show that the relation between production functions and separable cost functions is by no means as trivial as has been claimed.

Let $C(Y_1, Y_2)$ be the cost function of a firm which produces

Y_1 , and suffers an external diseconomy which is a function of Y_2 . That cost function is defined in terms of a production function $F(X_1 \dots X_n, Y_2)$, in the following manner:

$$C(Y_1, Y_2) = \min \sum_i W_i X_i$$

subject to $Y_1 = F(X_1 \dots X_n, Y_2)$.

Some precise definitions and lemmas regarding separability are needed:

Definition 1: A cost function $C(Y_1, Y_2)$ is separable if and only if it can be written as $C_1(Y_1) + C_2(Y_2)$.

Definition 2: A production function $F(X_1 \dots X_n, Y_2)$ is separable if and only if it can be written as $g(X_1 \dots X_n) + h(Y_2)$.

Lemma 1: A cost function is separable if and only if $\frac{\partial^2 C}{\partial Y_1 \partial Y_2} = 0$ everywhere.

Proof: Necessity is proved by differentiating $C = C_1 + C_2$ twice and observing that $\frac{\partial^2 C}{\partial Y_1 \partial Y_2} = 0$. Sufficiency is proved¹ by observing that the general solution of the second-order partial differential equation $\frac{\partial^2 C}{\partial Y_1 \partial Y_2} = 0$ is of the form $C = C_1(Y_1) + C_2(Y_2)$.

1. Lester Ford, [1955], p. 251 derives this result.

We assume throughout that C and F are continuously twice differentiable. The problem is to find what general class of functions F

give rise to cost functions with the property $\frac{\partial^2 C}{\partial Y_1 \partial Y_2} \equiv 0$. It has been claimed by Marchand and Russell that a cost function is separable if and only if the production function from which it is derived is separable. This conjecture is false in both directions. We begin by giving a simple example of a separable production function which does not generate a separable cost function. It will be seen later that the class of separable functions does not exhaust the class of functions giving rise to separable cost functions.

Consider a production function of the form

$$F = X_1^a X_2^\beta - C Y_2 \quad (1)$$

where $a + \beta < 1$. We find the cost function by solving the cost-minimization problem and using the first-order conditions and the production function to eliminate the inputs from the cost equation. From the first-order conditions we have

$$\frac{W_1}{W_2} = \frac{a X_2}{\beta X_1} \quad (2)$$

Solving for X_1 and substituting in (1) gives

$$Y_1 = X_2^{a+\beta} \left(\frac{W_2 a}{W_1 \beta} \right)^a - C Y_2 \quad (3)$$

Solving (3) for X_2 , and substituting the resulting expression for X_2 in (2) enables us to express X_1 and X_2 in terms of Y_1 and Y_2 alone. Substitution in $C = W_1 X_1 + W_2 X_2$ gives the cost function

$$C = W_2 \left(1 + \frac{a}{\beta} \right) \left(\frac{W_2 a}{W_1 \beta} \right)^{\frac{a}{a+\beta}} (Y_1 + C Y_2)^{\frac{1}{a+\beta}} \quad (4)$$

Clearly (4) is not separable if $a + \beta \neq 1$.

To find a production function which does generate a separable cost function we express $\frac{\partial^2 C}{\partial Y_1 \partial Y_2}$ in terms of the derivatives of the production function, and then find a general solution of the partial differential equation which results when $\frac{\partial^2 C}{\partial Y_1 \partial Y_2}$ is set equal to zero.

The general relation between cost and production functions is found by adopting the approach of Samuelson's Foundations. Consider the constrained cost minimization problem

$$\text{Minimize } \sum_i W_i X_i \text{ subject to } Y_1 - F(X_1, \dots, X_n) = 0.$$

We adopt the following abbreviations

$$\begin{aligned} \frac{\partial F}{\partial X_i} &= F_i & \frac{\partial F}{\partial Y} &= F_Y \\ \frac{\partial^2 F}{\partial X_i \partial X_j} &= F_{ij} & \frac{\partial^2 F}{\partial X_i \partial Y} &= F_{iY} \end{aligned}$$

Form the Lagrangian expression

$$L = \sum_i W_i X_i + \lambda (Y_1 - F(X_1, \dots, X_n, Y_2))$$

First-order conditions are

$$W_i - \lambda F_i = 0$$

$$Y_1 - F = 0.$$

We perturb the solution by varying Y_1 and Y_2 . Totally differentiating the first-order conditions gives the system of equations

$$\begin{pmatrix} F_{11} & \dots & F_{1n} & F_1 \\ \vdots & & \vdots & \vdots \\ F_{n1} & \dots & F_{nn} & F_n \\ F_1 & \dots & F_n & 0 \end{pmatrix} \begin{pmatrix} dX_1 \\ \vdots \\ dX_n \\ d\lambda/\lambda \end{pmatrix} = \begin{pmatrix} \frac{dW_1}{\lambda} - F_{1Y} dY_2 \\ \vdots \\ \frac{dW_n}{\lambda} - F_{nY} dY_2 \\ dY_1 - F_Y dY_2 \end{pmatrix} \quad (6)$$

$$\text{Let } \Delta = \begin{vmatrix} F_{11} & \dots & F_{1n} & F_1 \\ \vdots & & \vdots & \vdots \\ F_{n1} & \dots & F_{nn} & F_n \\ F_1 & \dots & F_n & 0 \end{vmatrix}$$

Further let Δ_{ij} be the i, j th co-factor of Δ .

$$\text{Since } C = \sum_k W_k X_k, \quad \frac{\partial C}{\partial Y_2} = \sum_k W_k \frac{\partial X_k}{\partial Y_2}.$$

Solving for dX_k using Cramer's rule gives

$$dX_k = \frac{\sum_{i=1}^n \left[\left(\frac{dW_i}{\lambda} - F_{iY} dY_2 \right) \Delta_{ik} + \left(dY_1 - F_Y dY_2 \right) \Delta_{n+1,k} \right]}{\Delta}$$

We assume that F is strictly quasi-concave in $X_1 \dots X_n$, so that $\Delta \neq 0$.

$$\text{Then } \frac{\partial X_k}{\partial Y_2} = \frac{-\sum_i F_{iY} \Delta_{ik} - F_Y \Delta_{n+1,k}}{\Delta},$$

$$\text{and } \frac{\partial C}{\partial Y_2} = \frac{-\sum_k W_k \left[\sum_i F_{iY} \Delta_{ik} + F_Y \Delta_{n+1,k} \right]}{\Delta}.$$

Since $W_k = \lambda F_k$,

$$\frac{\partial C}{\partial Y_2} = \frac{-\sum_{i=1}^n \left[F_{iY} \lambda \left(\sum_k F_k \Delta_{ik} \right) \right] - \lambda F_Y \sum_k F_k \Delta_{n+1,k}}{\Delta}.$$

But $\sum_k F_k \Delta_{n+1,k} = \Delta$, and $\sum_k F_k \Delta_{ik} = 0$ since it is an expansion by alien co-factors. Therefore

$$\frac{\partial C}{\partial Y_2} = -\lambda F_Y. \quad (7)$$

Differentiating (7) with respect to Y_1 gives

$$\frac{\partial^2 C}{\partial Y_2 \partial Y_1} = -\lambda \sum_i F_{iY} \frac{\partial X_i}{\partial Y_1} - F_Y \frac{\partial \lambda}{\partial Y_1}.$$

From (6),

$$\frac{\partial \lambda}{\partial Y_1} = \lambda \frac{\Delta_{n+1,n+1}}{\Delta}, \quad \frac{\partial X_i}{\partial Y_1} = \frac{\Delta_{n+1,i}}{\Delta}.$$

Therefore

$$\frac{\partial^2 C}{\partial Y_2 \partial Y_1} = -\frac{\lambda}{\Delta} \left(\sum_i F_{iY} \Delta_{n+1,i} + \Delta_{n+1,n+1} F_Y \right).$$

Characterizing the class of production functions which generate separable cost functions reduces to finding the general form of the solution of

$$\sum_i F_{iY} \Delta_{n+1,i} + \Delta_{n+1,n+1} F_Y = 0. \quad (8)$$

Note first that if $\Delta_{n+1,n+1} = 0$ and $F_{iY} = 0$ for all i , then (8) is satisfied.

Theorem 1: If F is of the form $g(X_1 \dots X_n) + h(Y_2)$ where g is homogeneous of degree one, then F generates a separable cost function.

Proof: Obviously $F_{iY} = 0$ for all i . If g is homogeneous of degree one then $[g_{ij}] = \Delta_{n+1,n+1} = 0$ wherever evaluated [Quirk and Saposnik, 1963]. Therefore F satisfies (8).

We begin by finding a general solution for (8) when there is just one input, denoted X . Then (8) becomes

$$F_X F_{YX} - F_Y F_{XX} = 0. \quad (9)$$

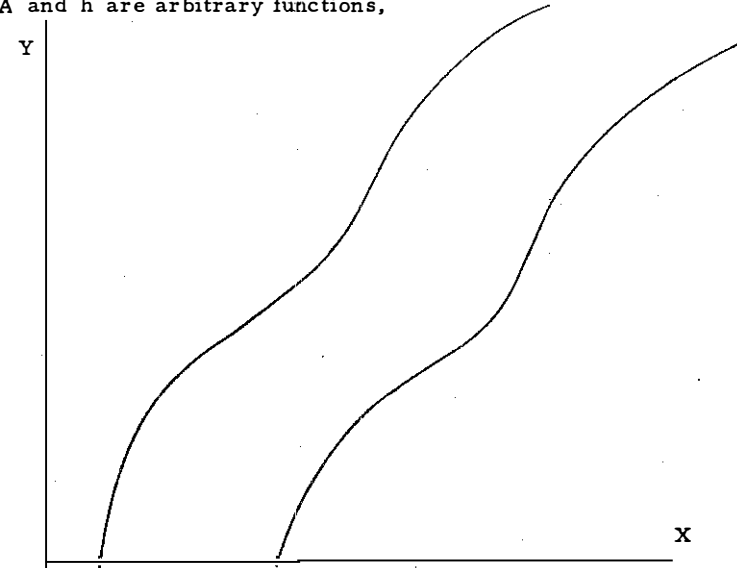
(9) is equal to the numerator of the expression $\frac{\partial}{\partial X} \left(\frac{F_Y}{F_X} \right)$. Thus the solution of (9) will be a function such that the ratio of F_Y to F_X is independent of X .

$$\text{Let } \frac{F_Y}{F_X} = -\phi(Y).$$

Then for any fixed value of F ,

$$\frac{dX}{dY} = \phi(Y).$$

In Figure 1, these level surfaces are illustrated. Each is simply a horizontal displacement of some other level surface (isoquant). Denote each isoquant $C(\lambda)$. Then on any isoquant $X = \lambda + f(Y)$ where f is some arbitrary function. The function F which solves (9) is an arbitrary function of λ , say $F = A(\lambda)$. Thus $F = A(X + h(Y))$, where A and h are arbitrary functions,



which may be restricted to preserve the convexity and strict quasi-concavity of X^2 . We check that this solution works by differentiating:

$$F_X = A'$$

$$F_Y = h' A'$$

$$\frac{F_Y}{F_X} = h'(Y).$$

2.

I am indebted to Joel Franklin for this demonstration.

This demonstration suggests that the solution to (8) may have the form

$$A(g(X_1 \dots X_n) + h(Y_2))$$

where A and h are arbitrary functions and $|g_{ij}| = 0$. We demonstrate that this is the case by establishing a relation between co-factors of

$$\begin{vmatrix} A_{ij} & A_i \\ A_j & 0 \end{vmatrix} = |A|$$

and co-factors of

$$\begin{vmatrix} g_{ij} & g_i \\ g_j & 0 \end{vmatrix} = |G|.$$

It is well known that $|A| = (A')^{n+1}|G|$.

Write

$$\begin{pmatrix} A_{ij} & A_i \\ A_i & 0 \end{pmatrix} = \begin{pmatrix} A''g_1^2 + A'g_{11} \dots A''g_1g_n + A'g_{1n} & A'g_1 \\ \vdots & \vdots \\ A''g_ng_1 + A'g_{n1} \dots A''g_n^2 + A'g_{nn} & A'g_n \\ A'g_1 & \dots & A'g_n & 0 \end{pmatrix}.$$

Then (8) becomes

$$\sum_i A'' h' g_i |A|_{n+1,i} + A' h' |A|_{n+1,n+1} = 0.$$

Since $\sum_i A' g_i |A|_{n+1,i} = |A|$, we have, if $h' \neq 0$,

$$\frac{A''}{A'} |A| + A' |A|_{n+1,n+1} = 0. \quad (10)$$

Lemma 3: If

$$C = \begin{pmatrix} C_{ij} & \vdots & C_i \\ \vdots & \ddots & \vdots \\ C_j & \vdots & 0 \end{pmatrix},$$

then

$$|C| = -\sum_{i,j} C_i C_j |C_{ij}|_{i,j}$$

where $|C_{ij}|_{i,j}$ is the co-factor of C_{ij} in the co-factor $|C|_{n+1,n+1}$.

Proof: Expand $|C|$ by its last column, obtaining

$$|C| = \sum_i C_i |C|_{i,n+1}$$

$$= C_1 \begin{vmatrix} C_{21} & \dots & C_{2n} \\ \vdots & & \vdots \\ C_{n1} & \dots & C_{nn} \\ C_1 & \dots & C_n \end{vmatrix} (-1)^{n+2} + \dots + C_n \begin{vmatrix} C_{11} & \dots & C_{1n} \\ \vdots & & \vdots \\ C_{n-1} & \dots & C_{n-1,n} \\ C_1 & \dots & C_n \end{vmatrix} (-1)^{2n+1}.$$

Now expand each determinant in this sum by its last row, obtaining

$$\begin{aligned}
 |C| &= C_1 C_1 \begin{vmatrix} C_{22} & \dots & C_{2n} \\ \vdots & & \vdots \\ C_{n2} & \dots & C_{nn} \end{vmatrix} (-1)^{n+2} (-1)^{n+1} + \dots + C_1 C_n \begin{vmatrix} C_{21} & \dots & C_{2,n-1} \\ \vdots & & \vdots \\ C_{n1} & \dots & C_{n,n-1} \end{vmatrix} (-1)^{n+2} (-1)^{2n} \\
 &+ C_n C_1 \begin{vmatrix} C_{12} & \dots & C_{1n} \\ \vdots & & \vdots \\ C_{n-1,2} & \dots & C_{n-1,n} \end{vmatrix} (-1)^{2n+1} (-1)^{n+1} + \dots + C_n C_n \begin{vmatrix} C_{11} & \dots & C_{1,n-1} \\ \vdots & & \vdots \\ C_{n-1,1} & \dots & C_{n-1,n-1} \end{vmatrix} (-1)^{2n+1} (-1)^{2n} \\
 &= \sum_i \sum_j C_i C_j \begin{vmatrix} C_{11} & \dots & C_{1,j-1} & C_{1,j+1} & \dots & C_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{i-1,1} & & & & & C_{i-1,n} \\ C_{i+1,1} & & & & & C_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ C_n & \dots & \dots & \dots & \dots & C_{nn} \end{vmatrix} (-1)^{2n+1+i+j}.
 \end{aligned}$$

Now

$$|C_{ij}|_{ij} = (-1)^{i+j} \begin{vmatrix} C_{11} & \dots & C_{1,j-1} & C_{1,j+1} & \dots & C_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{i-1,1} & & & & & C_{i-1,n} \\ C_{i+1,1} & & & & & C_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{n,1} & \dots & C_{n,j-1} & C_{n,j+1} & \dots & C_{nn} \end{vmatrix}.$$

Therefore

$$|C| = - \sum_i \sum_j C_i C_j |C_{ij}|_{i,j}.$$

Quirk and Ruppert [1968] have shown that for a matrix of the form of A, using the rule for evaluating the determinant of the sum of two matrices gives

$$|A|_{n+1,n+1} = (A')^n |G|_{n+1,n+1} + A'' (A')^{n-1} \sum_{j=1}^n g_j |G|_{n+1,j}.$$

Using our lemma,

$$\begin{aligned}
 |A|_{n+1,n+1} &= (A')^n |G|_{n+1,n+1} + A'' (A')^{n-1} \sum_j g_j |G|_{n+1,j} \\
 &= (A')^n |G|_{n+1,n+1} - A'' (A')^{n-1} |G| \\
 &= (A')^n |G|_{n+1,n+1} - A'' (A')^{-2} |A|.
 \end{aligned}$$

Therefore (10) becomes

$$\frac{A''}{A'} |A| + (A')^{n+1} |G|_{n+1,n+1} - \frac{A''}{A'} |A| = 0. \quad (11)$$

But $|G|_{n+1,n+1} = 0$ by hypothesis, so that the left hand side of (11) vanishes identically. The expression $A(g(X) + h(Y))$, containing two arbitrary functions, is the general solution of the differential equation (8).

If $F = A(g(X) + h(Y))$ where $|g_{ij}| = 0$, and if A and h are suitably restricted to preserve the quasi-concavity and concavity of F_n in X, then the resulting cost function will be separable. Separability of the production function is neither necessary nor sufficient for separability of the cost function.

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